

NON-SPECTRAL PROBLEM FOR THE PLANAR SELF-AFFINE MEASURES

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ABSTRACT. In this paper, we consider the non-spectral problem for the planar self-affine measures $\mu_{M,D}$ generated by an expanding integer matrix $M \in M_2(\mathbb{Z})$ and a finite digit set $D \subset \mathbb{Z}^2$. Let $p \geq 2$ be a positive integer, $E_p^2 := \frac{1}{p}\{(i, j)^t : 0 \leq i, j \leq p-1\}$ and $\mathcal{Z}_D^2 := \{x \in [0, 1)^2 : \sum_{d \in D} e^{2\pi i \langle d, x \rangle} = 0\}$. We show that if $\emptyset \neq \mathcal{Z}_D^2 \subset E_p^2 \setminus \{0\}$ and $\gcd(\det(M), p) = 1$, then there exist at most p^2 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$. In particular, if p is a prime, then the number p^2 is the best.

1. Introduction

Let $M \in M_n(\mathbb{R})$ be an $n \times n$ expanding real matrix (that is, all the eigenvalues of M have moduli > 1), and $D \subset \mathbb{R}^n$ be a finite subset with cardinality $\#(D)$. Let $\{\phi_d(x)\}_{d \in D}$ be an iterated function system (IFS) defined by

$$\phi_d(x) = M^{-1}(x + d) \quad (x \in \mathbb{R}^n, \quad d \in D).$$

Then the IFS arises a natural *self-affine measure* μ satisfying

$$\mu = \mu_{M,D} = \frac{1}{\#(D)} \sum_{d \in D} \mu \circ \phi_d^{-1}. \quad (1.1)$$

Such a measure $\mu_{M,D}$ is supported on the attractor $T(M, D)$ of the IFS $\{\phi_d\}_{d \in D}$ [8].

For a countable subset $\Lambda \subset \mathbb{R}^n$, let $\mathcal{E}_\Lambda = \{e^{2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda\}$. We call μ a spectral measure, and Λ a spectrum of μ if \mathcal{E}_Λ is an orthogonal basis for $L^2(\mu)$. We also say that (μ, Λ) is a *spectral pair*. The existence of a spectrum for μ is a basic problem in harmonic analysis, it was initiated by Fuglede in his seminal paper [7]. The first example of a singular, non-atomic, spectral measure was given by Jorgensen and Pedersen in [10]. This surprising discovery has received a lot of attention, and the research on the spectrality of self-affine measures has become an interesting topic. Also, new spectral measures were found in

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[9], [1]-[6], [13]-[14] and references cited therein. A related problem is the non-spectral problem of self-affine measure. In [4], Dutkay and Jorgensen showed that if $M = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}$ with $p \in \mathbb{Z} \setminus 3\mathbb{Z}$, $p \geq 2$ and

$$\mathcal{D} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \quad (1.2)$$

then there are no 4 mutually orthogonal exponential functions in $L^2(\mu_{M,\mathcal{D}})$; they also prove that if $M = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, then there exist at most 7 mutually orthogonal exponential functions in $L^2(\mu_{M,\mathcal{D}})$. In [11], the third author of this paper proved that if the expanding integer matrix $M = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ with $ac \notin 3\mathbb{Z}$, then there exist at most 3 mutually orthogonal exponential functions in $L^2(\mu_{M,\mathcal{D}})$, and the number 3 is the best. The third author also obtained the same conclusions for the expanding integer matrix $M = \begin{bmatrix} a & b \\ d & c \end{bmatrix}$ with $\det(M) = ac - bd \notin 3\mathbb{Z}$ in [12]. In this paper, we will give a more general result which is suitable for more self-affine measures. Before the statement of the main results, we first give some definitions and notations.

For a positive integer $p \geq 2$ and a finite digit set $D \subset \mathbb{Z}^n$, let

$$E_p^n := \frac{1}{p} \{(l_1, l_2, \dots, l_n)^t : 0 \leq l_1, \dots, l_n \leq p-1\}, \quad \mathring{E}_p^n := E_p^n \setminus \{0\} \quad (1.3)$$

and

$$m_D(x) = \frac{1}{\#(D)} \sum_{d \in D} e^{2\pi i \langle d, x \rangle}, \quad x \in \mathbb{R}^n, \quad (1.4)$$

where $m_D(x)$ is called the *mask polynomial* of D as usual.

Define $\mathcal{Z}(m_D) := \{x \in \mathbb{R}^n : m_D(x) = 0\}$ and

$$\mathcal{Z}_D^n := \mathcal{Z}(m_D) \cap [0, 1)^n. \quad (1.5)$$

It is easy to see that m_D is a \mathbb{Z}^n -periodic function for $D \subset \mathbb{Z}^n$. In this case,

$$\mathcal{Z}(m_D) = \mathcal{Z}_D^n + \mathbb{Z}^n. \quad (1.6)$$

Theorem 1.1. *Let $p \geq 2$ be a positive integer, $D \subset \mathbb{Z}^2$ be a finite digit set, $M \in M_2(\mathbb{Z})$ be an expanding integer matrix, and let $\mu_{M,D}$, \mathring{E}_p^2 , \mathcal{Z}_D^2 be defined by (1.1), (1.3) and (1.5), respectively. If $\emptyset \neq \mathcal{Z}_D^2 \subset \mathring{E}_p^2$ and $\gcd(\det(M), p) = 1$, then there exist at most p^2 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$. In particular, if p is a prime, then the number p^2 is the best.*

In fact, we can prove a more general result. In order to state this conclusion, we need the following definition.

Definition 1.2. Let μ be a Borel probability measure with compact support on \mathbb{R}^n . Let Λ be a finite or countable subset of \mathbb{R}^n , and let $\mathcal{E}_\Lambda = \{e^{2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda\}$. We denote \mathcal{E}_Λ by \mathcal{E}_Λ^* if \mathcal{E}_Λ is a maximal orthogonal set of exponential functions in $L^2(\mu)$. Let

$$n^*(\mu) := \sup\{\#\Lambda : \mathcal{E}_\Lambda^* \text{ is a maximal orthogonal set}\}, \quad (1.7)$$

and call $n^*(\mu)$ the maximal cardinality of the orthogonal exponential functions in $L^2(\mu)$.

Obviously, \mathcal{E}_Λ^* is not unique, even among those Λ with equal cardinality. For a positive integer $p \geq 2$, let \mathfrak{M}_p denote the class of all expanding matrices $M \in M_2(\mathbb{Z})$ with $\gcd(\det(M), p) = 1$. If $D \subset \mathbb{Z}^2$ is a finite digit set with $\emptyset \neq \mathcal{Z}_D^2 \subset \mathring{E}_p^2$, then $n^*(\mu_{M,D}) \leq p^2$ for $M \in \mathfrak{M}_p$ by Theorem 1.1. Furthermore, we can divide (see Theorem 1.3 below) \mathfrak{M}_p into two disjoint subclasses $\mathfrak{M}_p^{(1)}$ and $\mathfrak{M}_p^{(2)}$ such that $n^*(\mu_{M,D}) \leq \widetilde{P}$ for $M \in \mathfrak{M}_p^{(1)}$, and $n^*(\mu_{M,D}) = p^2$ for $M \in \mathfrak{M}_p^{(2)}$, where

$$\widetilde{P} = \begin{cases} \frac{p^2}{2}, & \text{if } p \text{ is even,} \\ \frac{p^2-1}{2}, & \text{if } p > 3 \text{ is odd,} \\ 3, & \text{if } p = 3. \end{cases} \quad (1.8)$$

For $M \in \mathfrak{M}_p$, we denote the transposed conjugate of M by M^* . Let $\lambda \in \mathring{E}_p^2$. We observe (see Proposition 2.2) that $M^{*j}\lambda \in \mathring{E}_p^2 \pmod{\mathbb{Z}^2}$ for each j . From this conclusion, we can easily prove $\bigcup_{j=1}^\infty \{M^{*j}\lambda\} = \bigcup_{j=1}^{p^2-1} \{M^{*j}\lambda\} \pmod{\mathbb{Z}^2}$, since there exist only $p^2 - 1$ elements in \mathring{E}_p^2 . Let

$$\begin{cases} \mathfrak{M}_p^{(1)} = \{M : \bigcup_{j=1}^{p^2-1} M^{*j}\mathcal{Z}_D^2 \subsetneq \mathring{E}_p^2 \pmod{\mathbb{Z}^2}\}, \\ \mathfrak{M}_p^{(2)} = \{M : \bigcup_{j=1}^{p^2-1} M^{*j}\mathcal{Z}_D^2 = \mathring{E}_p^2 \pmod{\mathbb{Z}^2}\}. \end{cases} \quad (1.9)$$

It is possible that $\mathfrak{M}_p^{(1)} = \emptyset$ (see Example 4.3), but if p is a prime, then $\mathfrak{M}_p^{(2)} \neq \emptyset$ (see Proposition 2.6).

Theorem 1.3. Under the conditions of Theorem 1.1, let $n^*(\mu_{M,D})$ be given by (1.7). Then

$$n^*(\mu_{M,D}) \begin{cases} \leq \widetilde{P}, & \text{if } M \in \mathfrak{M}_p^{(1)}, \\ = p^2, & \text{if } M \in \mathfrak{M}_p^{(2)}, \end{cases}$$

where \widetilde{P} and $\mathfrak{M}_p^{(1)}$, $\mathfrak{M}_p^{(2)}$ are defined by (1.8) and (1.9), respectively.

By Theorem 1.3, to prove Theorem 1.1, we only need to prove that if p is a prime, then $\mathfrak{M}_p^{(2)} \neq \emptyset$. In particular, if $p = 2$ or 3 , we can get a more stronger result.

Theorem 1.4. Under the conditions of Theorem 1.3, if $p = 2$ or 3 , then

$$n^*(\mu_{M,D}) = \begin{cases} p, & \text{if } M \in \mathfrak{M}_p^{(1)}, \\ p^2, & \text{if } M \in \mathfrak{M}_p^{(2)}. \end{cases}$$

The organization of the paper is as follows. We give several preparatory definitions and conclusions in Section 2, and prove Theorems 1.1, 1.3 and 1.4 in Section 3. As an application, in Section 4 we give a complete discussion for the Sierpinski measure in [12] and construct an example to illustrate some special case.

2. Preliminaries

In this section, we give some preliminary definitions and propositions. We will start with an introduction to the Fourier transform. Let $M \in M_n(\mathbb{R})$ be an $n \times n$ expanding real matrix, $D \subset \mathbb{R}^n$ be a finite subset with cardinality $|D|$. Let $\mu_{M,D}$ be defined by (1.1). In the study of spectrality of $\mu_{M,D}$, the Fourier transform $\hat{\mu}_{M,D}(\xi) = \int e^{2\pi i \langle x, \xi \rangle} d\mu_{M,D}(x)$, ($\xi \in \mathbb{R}^n$) of $\mu_{M,D}$ plays an important role. It follows from [4] that

$$\hat{\mu}_{M,D}(\xi) = \prod_{j=1}^{\infty} m_D(M^{*-j}\xi), \quad (\xi \in \mathbb{R}^n) \quad (2.1)$$

where M^* denotes the transposed conjugate of M , and

$$m_D(x) = \frac{1}{\#(D)} \sum_{d \in D} e^{2\pi i \langle d, x \rangle}, \quad (x \in \mathbb{R}^n).$$

For any $\lambda_1, \lambda_2 \in \mathbb{R}^n$, $\lambda_1 \neq \lambda_2$, the orthogonality condition

$$0 = \langle e^{2\pi i \langle \lambda_1, x \rangle}, e^{2\pi i \langle \lambda_2, x \rangle} \rangle_{L^2(\mu_{M,D})} = \int e^{2\pi i \langle \lambda_1 - \lambda_2, x \rangle} d\mu_{M,D}(x) = \hat{\mu}_{M,D}(\lambda_1 - \lambda_2)$$

relates to the zero set $\mathcal{Z}(\hat{\mu}_{M,D})$ directly. It is easy to see that for a countable subset $\Lambda \subset \mathbb{R}^n$, $\mathcal{E}_\Lambda = \{e^{2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda\}$ is an orthonormal family of $L^2(\mu_{M,D})$ if and only if

$$(\Lambda - \Lambda) \setminus \{0\} \subset \mathcal{Z}(\hat{\mu}_{M,D}). \quad (2.2)$$

From (2.1), we have $\mathcal{Z}(\hat{\mu}_{M,D}) = \{\xi \in \mathbb{R}^n : \exists j \in \mathbb{N} \text{ such that } m_D(M^{*-j}\xi) = 0\}$. Hence

$$\mathcal{Z}(\hat{\mu}_{M,D}) = \bigcup_{j=1}^{\infty} M^{*j}(\mathcal{Z}(m_D)), \quad (2.3)$$

where $\mathcal{Z}(m_D) = \{x \in \mathbb{R}^n : m_D(x) = 0\}$.

The following lemma can be used to determine the case that there are only finite orthogonal exponential functions in $L^2(\mu_{M,D})$.

Lemma 2.1. [4, Theorem 3.1] *For a $n \times n$ expanding integer matrix M and a finite digit set $D \subset \mathbb{Z}^n$, let $\mu_{M,D}$ and \mathcal{Z}_D^n be defined by (1.1) and (1.5), respectively. If \mathcal{Z}_D^n is contained in a set $Z' \subset [0, 1)^n$ of finite cardinality $\#(Z')$, which does not contain 0, and that satisfies the property*

$$M^*(Z' + \mathbb{Z}^n) \subseteq Z' + \mathbb{Z}^n,$$

then there exist at most $\#(\mathbb{Z}') + 1$ mutually orthogonal exponential functions in $L^2(\mu_{M,D})$. In particular, $\mu_{M,D}$ is not a spectral measure.

Proposition 2.2. *Let $p \geq 2$ be a positive integer and \mathring{E}_p^2 be defined by (1.3), and let $M \in M_2(\mathbb{Z})$ with $\gcd(\det(M), p) = 1$. Then $M(p\mathring{E}_p^2) = p\mathring{E}_p^2 \pmod{p\mathbb{Z}^2}$, equivalently, $M(\mathring{E}_p^2) = \mathring{E}_p^2 \pmod{\mathbb{Z}^2}$.*

Proof. We first prove $M(p\mathring{E}_p^2) \subset p\mathring{E}_p^2 \pmod{p\mathbb{Z}^2}$. Suppose on the contrary that there exist $(l_1, l_2)^t \in p\mathring{E}_p^2$ and $v_0 \in \mathbb{Z}^2$ such that

$$M \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = pv_0. \quad (2.4)$$

Multiplying $\det(M)p^{-1}M^{-1}$ on both sides of (2.4), we get

$$\frac{\det(M)}{p} \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \det(M)M^{-1}v_0.$$

Obviously, $\det(M)M^{-1}v_0 \in \mathbb{Z}^2$, but the left side of the above equation can not be a integer vector since $\gcd(\det(M), p) = 1$ and $(l_1, l_2)^t \in p\mathring{E}_p^2$, this contradiction shows that $M(p\mathring{E}_p^2) \subset p\mathring{E}_p^2 \pmod{p\mathbb{Z}^2}$.

We now prove $M(p\mathring{E}_p^2) = p\mathring{E}_p^2 \pmod{p\mathbb{Z}^2}$. For any $\lambda \neq \lambda' \in p\mathring{E}_p^2$, it follows from $(\lambda - \lambda') \in p\mathring{E}_p^2 \pmod{p\mathbb{Z}^2}$ and $M(p\mathring{E}_p^2) \subset p\mathring{E}_p^2 \pmod{p\mathbb{Z}^2}$ that $M\lambda \neq M\lambda' \pmod{p\mathbb{Z}^2}$. Hence $\#(M(p\mathring{E}_p^2) \pmod{p\mathbb{Z}^2}) = \#(p\mathring{E}_p^2)$, this gives that $M(p\mathring{E}_p^2) = p\mathring{E}_p^2 \pmod{p\mathbb{Z}^2}$ since $M(p\mathring{E}_p^2) \subset p\mathring{E}_p^2 \pmod{p\mathbb{Z}^2}$. It also shows that $M(\mathring{E}_p^2) = \mathring{E}_p^2 \pmod{\mathbb{Z}^2}$. \square

To prove that the number p^2 is the best in Theorem 1.1, we need the following knowledge related to the theory of number and algebra.

For a positive number m , let $\varphi(m)$ denote the *Euler's phi function* (also be called *Euler's totient function*) which equal to the number of integers in the set $\{1, 2, \dots, m-1\}$ that are relatively prime to m . For more information about the Euler's phi function, the reader can refer to [17]. The following lemma is the famous *Euler's theorem*.

Lemma 2.3. [17, Theorem 2.12] *Let m be a positive integer, and let a be an integer relatively prime to m . Then $a^{\varphi(m)} = 1 \pmod{m}$.*

For a prime p , let $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ denote the residue class fields. All nonsingular $n \times n$ matrices over \mathbb{F}_p form a finite group under matrix multiplication, called the *general linear group* $GL(n, \mathbb{F}_p)$.

Definition 2.4. *Let $f(x) \in \mathbb{F}_p[x]$ be a nonzero polynomial. If $f(0) \neq 0$, then the least positive integer q for which $f(x)$ divides $x^q - 1$ is called the order of f and denoted by $\text{ord}(f)$.*

Definition 2.5. Let $M \in GL(n, \mathbb{F}_p)$, then the least positive integer s for which $M^s = I$ is called the order of M and denoted by $O(M)$, where I is the identity matrix in $GL(n, \mathbb{F}_p)$.

Let $f(x) = x^n - a_{n-1}x^{n-1} - \cdots - a_1x - a_0 \in \mathbb{F}_p[x]$ and $f(0) \neq 0$, the companion matrix M of $f(x)$ is defined by

$$M := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & 0 & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & a_{n-1} \end{bmatrix}.$$

It follows from Theorem 6.26 in [16] that $M \in GL(n, \mathbb{F}_p)$ and $\text{ord}(f) = O(M)$. Further, for any positive integer $n \geq 1$, if $f(x) \in \mathbb{F}_p[x]$ is a primitive polynomial of degree n , then by Theorem 3.16 in [16], we know that f is monic, $f(0) \neq 0$, and $\text{ord}(f) = p^n - 1$. It is well known that there exist $\varphi(p^n - 1)/n$ primitive polynomials with degree n over \mathbb{F}_p (see P_{87} Theorem 4.1.3 of [15]), where φ is Euler's phi function.

The following Proposition can be proved more easily by using the group and orbit theory, however, we will prove it directly by avoiding to introduce more definitions and notations.

Proposition 2.6. Let p be a prime, then there exists a matrix $M \in M_2(\mathbb{Z})$ with $\gcd(\det(M), p) = 1$ such that

$$\bigcup_{j=1}^{p^2-1} \{M^j \lambda\} = \mathring{E}_p^2 \pmod{\mathbb{Z}^2} \quad \text{for all } \lambda \in \mathring{E}_p^2. \quad (2.5)$$

Proof. Let $f(x) = x^2 - a_1x - a_0 \in \mathbb{F}_p[x]$ be a primitive polynomial and $M = \begin{bmatrix} 0 & a_0 \\ 1 & a_1 \end{bmatrix}$ be the companion matrix of $f(x)$, then $M \in GL(2, \mathbb{F}_p)$ and $\text{ord}(f) = O(M) = p^2 - 1$. Moreover, $\gcd(\det(M), p) = \gcd(-a_0, p) = 1$ because $f(0) \neq 0$ and p is prime.

In order to prove the matrix M satisfies (2.5), we first need to prove

$$\left\{ M \frac{e_1}{p}, M^2 \frac{e_1}{p}, \dots, M^{p^2-1} \frac{e_1}{p} \right\} = \mathring{E}_p^2 \pmod{\mathbb{Z}^2}, \quad (2.6)$$

where $e_1 = (1, 0)^t$. Assume that the (2.6) has been proved. For any $\lambda \in \mathring{E}_p^2$, (2.6) implies that there exist $k_0 \in \{1, \dots, p^2 - 1\}$ and $v_0 \in \mathbb{Z}^2$ such that $M^{k_0} \frac{e_1}{p} = \lambda + v_0$. By $\gcd(\det(M), p) = 1$ and Proposition 2.2, we have

$$\begin{aligned} \mathring{E}_p^2 &= M^{k_0}(\mathring{E}_p^2) = M^{k_0}(\{M \frac{e_1}{p}, M^2 \frac{e_1}{p}, \dots, M^{p^2-1} \frac{e_1}{p}\}) \pmod{\mathbb{Z}^2} \\ &= \{M(\lambda + v_0), M^2(\lambda + v_0), \dots, M^{p^2-1}(\lambda + v_0)\} \pmod{\mathbb{Z}^2} \\ &= \{M\lambda, M^2\lambda, \dots, M^{p^2-1}\lambda\} \pmod{\mathbb{Z}^2}. \end{aligned}$$

This indicates that M is exactly what we need.

We now prove (2.6). Let $e_2 = (0, 1)$, and write

$$k_1 = \min\{k : M^k e_1 = e_1 \pmod{p\mathbb{Z}^2}\}, \quad k_2 = \min\{k : M^k e_2 = e_2 \pmod{p\mathbb{Z}^2}\}.$$

Obviously, $1 \leq k_1, k_2 \leq p^2 - 1$ as $O(M) = p^2 - 1$. We conclude

$$\{Me_1, \dots, M^{k_1}e_1\} \pmod{p\mathbb{Z}^2} \text{ contains exact } k_1 \text{ different elements.} \quad (2.7)$$

Suppose, on the contrary, that there exist $1 \leq l < l' \leq k_1$ and $v_0 \in \mathbb{Z}^2$ such that $M^{l'}e_1 = M^l e_1 + pv_0$, i.e., $M^l(M^{l'-l}e_1 - e_1) = pv_0$. Noting that $(M^{l'-l}e_1 - e_1) \in p\mathring{E}_p^2 \pmod{p\mathbb{Z}^2}$, by Proposition 2.2, we have $M^{l'-l}e_1 - e_1 = 0 \pmod{p\mathbb{Z}^2}$, which contradicts with the definition of k_1 . Hence (2.7) holds. Similarly, we have

$$\{Me_2, \dots, M^{k_2}e_2\} \pmod{p\mathbb{Z}^2} \text{ contains exact } k_2 \text{ different elements.} \quad (2.8)$$

We further claim that

$$k_1 = k_2 = p^2 - 1. \quad (2.9)$$

If $k_1 < k_2$, then by using $Me_1 = e_2$ (a directly check), we have $M^{k_1+1}e_2 = Me_2$, this contradicts with (2.8). If $k_1 > k_2$, then $M^{k_2+1}e_1 = Me_1$, this contradicts with (2.7). So the claim $k_1 = k_2$ follows. From the definitions of k_1, k_2 , we have $M^{k_1}\{e_1, e_2\} = \{e_1, e_2\} \pmod{p\mathbb{Z}^2}$, hence $M^{k_1}I = M^{k_1} = I \pmod{pM_2(\mathbb{Z})}$ which shows (2.9) holds since $k_1 \leq p^2 - 1$ and $O(M) = p^2 - 1$.

It follows from Proposition 2.2 that $\{Me_1, M^2e_1, \dots, M^{p^2-1}e_1\} \subset p\mathring{E}_p^2 \pmod{p\mathbb{Z}^2}$. Since $\{Me_1, M^2e_1, \dots, M^{p^2-1}e_1\} \pmod{p\mathbb{Z}^2}$ contains exact $p^2 - 1$ different elements by (2.7) and (2.9), and since $\#(p\mathring{E}_p^2) = p^2 - 1$, we get $\{Me_1, M^2e_1, \dots, M^{p^2-1}e_1\} = p\mathring{E}_p^2 \pmod{p\mathbb{Z}^2}$, and the required result (2.6) holds.

This completes the proof of Proposition 2.6. \square

To prove the Theorems, we also need the following proposition.

Proposition 2.7. *For a positive integer $p \geq 2$, let E_p^2 and \widetilde{P} be defined by (1.3) and (1.8), respectively. Let $m = \widetilde{P} + 1$ and $A = \{\lambda_1, \lambda_2, \dots, \lambda_m\} \subset E_p^2$ with $\lambda_i \neq \lambda_j$ for any $i \neq j$. Then $A - A = E_p^2 \pmod{\mathbb{Z}^2}$.*

Proof. Obviously, $(A - A) \subseteq E_p^2 \pmod{\mathbb{Z}^2}$. If we suppose, on the contrary, that $(A - A) \subsetneq E_p^2 \pmod{\mathbb{Z}^2}$, then there exists $\tau \in \mathring{E}_p^2$ satisfies $\tau \notin (A - A) \pmod{\mathbb{Z}^2}$. We claim that $-\tau \notin (A - A) \pmod{\mathbb{Z}^2}$, if not, there are $1 \leq i_0 \neq j_0 \leq m$ and $v_0 \in \mathbb{Z}^2$ such that $-\tau = \lambda_{i_0} - \lambda_{j_0} + v_0$. Hence $\tau = \lambda_{j_0} - \lambda_{i_0} - v_0$, this implies that $\tau \in (A - A) \pmod{\mathbb{Z}^2}$, which contradicts with $\tau \notin (A - A) \pmod{\mathbb{Z}^2}$. So the claim $-\tau \notin (A - A) \pmod{\mathbb{Z}^2}$ follows.

Let $\beta_i = \lambda_i - \tau$, $\gamma_i = \lambda_i + \tau$ for $i = 1, 2, \dots, m$. Then for each $i \in \{1, 2, \dots, m\}$, we have $\beta_i, \gamma_i \notin A \pmod{\mathbb{Z}^2}$. Otherwise if $\beta_{i_0} \in A \pmod{\mathbb{Z}^2}$, then $\beta_{i_0} - \lambda_{i_0} \in (A - A) \pmod{\mathbb{Z}^2}$,

which contradicts with $\beta_{i_0} - \lambda_{i_0} = -\tau \notin (A - A) \pmod{\mathbb{Z}^2}$. Similarly, $\gamma_{i_0} \in A \pmod{\mathbb{Z}^2}$ also yields a contradiction. Thus we get

$$\{\beta_1, \dots, \beta_m\} \cup \{\gamma_1, \dots, \gamma_m\} \subset E_p^2 \setminus A \pmod{\mathbb{Z}^2}. \quad (2.10)$$

Noting that $\beta_i - \beta_j = \gamma_i - \gamma_j = \lambda_i - \lambda_j \neq 0 \pmod{\mathbb{Z}^2}$ for $i \neq j$, we have

$$\beta_i \neq \beta_j \pmod{\mathbb{Z}^2} \quad \text{and} \quad \gamma_i \neq \gamma_j \pmod{\mathbb{Z}^2} \quad \text{if} \quad i \neq j. \quad (2.11)$$

(i) If $p \neq 3$, by the definition of \widetilde{P} in (1.8), we have $m = \frac{p^2}{2} + 1$ when p is even and $m = \frac{p^2-1}{2} + 1$ when p is odd. By (2.10)-(2.11), we have $m = \#\{\beta_1, \dots, \beta_m\} \pmod{\mathbb{Z}^2} \leq \#(E_p^2 \setminus A) = p^2 - m$, i.e., $2m \leq p^2$, this is impossible. Hence $(A - A) = E_p^2 \pmod{\mathbb{Z}^2}$.

(ii) If $p = 3$, then $m = 4$. By noting that $\#(E_3^2 \setminus A) = 5$ and (2.10)-(2.11), there exist at least one pair of β_{i_1} and γ_{j_1} such that $\beta_{i_1} = \gamma_{j_1} \pmod{\mathbb{Z}^2}$, i.e., $\lambda_{i_1} - \tau = \lambda_{j_1} + \tau + v$, or $\lambda_{i_1} - \lambda_{j_1} - v = 2\tau$ for some $v \in \mathbb{Z}^2$. This is equivalent to $\lambda_{i_1} - \lambda_{j_1} = -\tau \pmod{\mathbb{Z}^2}$ since $3\tau \in \mathbb{Z}^2$, it contradicts with $-\tau \notin (A - A) \pmod{\mathbb{Z}^2}$. This shows that $(A - A) = E_3^2 \pmod{\mathbb{Z}^2}$. \square

3. The proofs of the main theorems

In this section, we first prove Theorem 1.3 by Propositions 2.2 and 2.7, and then prove Theorem 1.1 by applying Proposition 2.6. Finally we prove Theorem 1.4.

The proof of Theorem 1.3. (i) We first prove $n^*(\mu_{M,D}) \leq \widetilde{P}$ for $M \in \mathfrak{M}_p^{(1)}$, where \widetilde{P} is given by (1.8). We will prove this by contradiction. Before the proof, we first give some properties on the zeros of Fourier transform $\hat{\mu}_{M,D}$.

For any $\lambda \in \mathcal{Z}_D^2 \subset \mathring{E}_p^2$, by Proposition 2.2, we see that $M^{*j}\lambda \in \mathring{E}_p^2 \pmod{\mathbb{Z}^2}$ for any j . Note that \mathring{E}_p^2 has only $p^2 - 1$ different elements. Then $\bigcup_{j=1}^{\infty} M^{*j}\lambda = \bigcup_{j=1}^{p^2-1} M^{*j}\lambda \pmod{\mathbb{Z}^2}$, and so

$$\bigcup_{j=1}^{\infty} M^{*j}(\mathcal{Z}_D^2) = \bigcup_{j=1}^{p^2-1} M^{*j}(\mathcal{Z}_D^2) \pmod{\mathbb{Z}^2}. \quad (3.1)$$

Since $M \in \mathfrak{M}_p^{(1)}$ is an integer matrix, it follows from (2.3), (1.6), (3.1) and (1.9) that

$$\begin{aligned} \mathcal{Z}(\hat{\mu}_{M,D}) &= \bigcup_{j=1}^{\infty} M^{*j}(\mathcal{Z}(m_D)) = \bigcup_{j=1}^{\infty} M^{*j}(\mathcal{Z}_D^2 + \mathbb{Z}^2) \\ &\subset \left(\bigcup_{j=1}^{\infty} M^{*j}(\mathcal{Z}_D^2) \right) + \mathbb{Z}^2 = \left(\bigcup_{j=1}^{p^2-1} M^{*j}(\mathcal{Z}_D^2) \right) + \mathbb{Z}^2 \subset \mathring{E}_p^2 + \mathbb{Z}^2. \end{aligned} \quad (3.2)$$

Suppose on the contrary that there exists $\Lambda' = \{0, \lambda_1, \lambda_2, \dots, \lambda_{\widetilde{P}}\} \subset \mathbb{R}^2$ such that $\mathcal{E}_{\Lambda'} = \{e^{2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda'\}$ is a orthogonal set in $L^2(\mu_{M,D})$. The orthogonal condition implies that

$$(\Lambda' - \Lambda') \setminus \{0\} \subset \mathcal{Z}(\widehat{\mu}_{M,D}),$$

and hence $\lambda_1 - 0, \dots, \lambda_{\widetilde{P}} - 0 \in \mathring{E}_p^2 \pmod{\mathbb{Z}^2}$ by (3.2). Because $\mathcal{Z}(\widehat{\mu}_{M,D})$ does not contain any integer, we have $\lambda_j - \lambda_k \notin \mathbb{Z}^2$, i.e. $\lambda_j \not\equiv \lambda_k \pmod{\mathbb{Z}^2}$ for $j \neq k$. This yields that $A := \Lambda' \pmod{\mathbb{Z}^2}$ has $\widetilde{P} + 1$ different elements in E_p^2 . By Proposition 2.7, we have $A - A = E_p^2 \pmod{\mathbb{Z}^2}$, and hence

$$\mathring{E}_p^2 \subset \mathcal{Z}(\widehat{\mu}_{M,D}) \pmod{\mathbb{Z}^2}.$$

On the other hand, since $\bigcup_{j=1}^{p^2-1} M^{*j} \mathcal{Z}_D^2 \subsetneq \mathring{E}_p^2$ for $M \in \mathfrak{M}_p^{(1)}$, (2.3), (1.6) and (3.1) yield that

$$\mathcal{Z}(\widehat{\mu}_{M,D}) = \bigcup_{j=1}^{\infty} M^{*j}(\mathcal{Z}(m_D)) = \bigcup_{j=1}^{\infty} M^{*j}(\mathcal{Z}_D^2 + \mathbb{Z}^2) = \bigcup_{j=1}^{p^2-1} M^{*j} \mathcal{Z}_D^2 \subsetneq \mathring{E}_p^2 \pmod{\mathbb{Z}^2}.$$

This contradiction shows that such $\mathcal{E}_{\Lambda'}$ does not exist. Hence there exist at most \widetilde{P} mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, i.e., $n^*(\mu_{M,D}) \leq \widetilde{P}$.

(ii) We now prove that $n^*(\mu_{M,D}) = p^2$ for $M \in \mathfrak{M}_p^{(2)}$.

Firstly, we prove that $n^*(\mu_{M,D}) \leq p^2$. Since M is an integer matrix and $\gcd(\det(M), p) = 1$, by Proposition 2.2, we have $M(\mathring{E}_p^2 + \mathbb{Z}^2) = M(\mathring{E}_p^2) + M(\mathbb{Z}^2) \subset M(\mathring{E}_p^2) + \mathbb{Z}^2 = \mathring{E}_p^2 + \mathbb{Z}^2$. It follows from Proposition 2.1 and $\mathcal{Z}_D^2 \subset \mathring{E}_p^2$ that

$$n^*(\mu_{M,D}) \leq \#(\mathring{E}_p^2) + 1 = p^2. \quad (3.3)$$

Secondly, we prove that $n^*(\mu_{M,D}) \geq p^2$. We will prove it by finding out exact p^2 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$.

Let $\det(M) = L$ and $\varphi(p)$ denote the Euler phi function. It follows from $\gcd(L, p) = 1$ and Lemma 2.3 that there exist integer n such that

$$L^{\varphi(p)} = np + 1. \quad (3.4)$$

For any $\lambda \in \mathring{E}_p^2$, we first prove that

$$M^{*j}(\lambda + \mathbb{Z}^2) \supset L^{\varphi(p)j} (M^{*j} \lambda + \mathbb{Z}^2), \quad (3.5)$$

which is equivalent to

$$\lambda + \mathbb{Z}^2 \supset L^{\varphi(p)j} M^{*-j} (M^{*j} \lambda + \mathbb{Z}^2).$$

Taking an arbitrary point $v_1 \in \mathbb{Z}^2$, we have

$$L^{\varphi(p)j} M^{*-j} (M^{*j} \lambda + v_1) = L^{\varphi(p)j} \lambda + L^{\varphi(p)j} M^{*-j} v_1.$$

By (3.4), we have $L^{\varphi(p)j} = (np + 1)^j = pm + 1$ for some integer m . It follows from $L^{\varphi(p)j} M^{*-j} \in M_2(\mathbb{Z})$ and $pm\lambda \in \mathbb{Z}^2$ that

$$L^{\varphi(p)j} \lambda + L^{\varphi(p)j} M^{*-j} v_1 = \lambda + pm\lambda + L^{\varphi(p)j} M^{*-j} v_1 \in \lambda + \mathbb{Z}^2.$$

This shows that (3.5) holds. Hence, from (2.3), (1.6) and (3.5), we deduce that

$$\begin{aligned}\mathcal{Z}(\hat{\mu}_{M,D}) &= \bigcup_{j=1}^{\infty} M^{*j}(\mathcal{Z}(m_D)) = \bigcup_{j=1}^{\infty} M^{*j}(\mathcal{Z}_D^2 + \mathbb{Z}^2) \\ &\supset \bigcup_{j=1}^{\infty} L^{\varphi(p)j}(M^{*j}\mathcal{Z}_D^2 + \mathbb{Z}^2) \supset \bigcup_{j=1}^{p^2-1} L^{\varphi(p)j}(M^{*j}\mathcal{Z}_D^2 + \mathbb{Z}^2).\end{aligned}\quad (3.6)$$

Let $\Lambda = L^{\varphi(p)(p^2-1)}E_p^2$. We will show that $\mathcal{E}_\Lambda = \{e^{2\pi i\langle \lambda, x \rangle} : \lambda \in \Lambda\}$ is an orthogonal set in $L^2(\mu_{M,D})$. For any $\lambda_1 \neq \lambda_2 \in \Lambda$, there exists $\lambda' \in \mathring{E}_p^2 \pmod{\mathbb{Z}^2}$ such that $\lambda_1 - \lambda_2 = L^{\varphi(p)(p^2-1)}\lambda'$. Since $\bigcup_{j=1}^{p^2-1} M^{*j}\mathcal{Z}_D^2 = \mathring{E}_p^2 \pmod{\mathbb{Z}^2}$ for $M \in \mathfrak{M}_p^{(2)}$, there exist $\lambda_0 \in \mathcal{Z}_D^2$ and $j_0 \in \{1, \dots, p^2-1\}$ such that $\lambda' = M^{*j_0}\lambda_0 \pmod{\mathbb{Z}^2}$. Then

$$\lambda_1 - \lambda_2 \in L^{\varphi(p)(p^2-1)}(M^{*j_0}\lambda_0 + \mathbb{Z}^2) = L^{\varphi(p)j_0}(L^{\varphi(p)(p^2-1-j_0)}(M^{*j_0}\lambda_0 + \mathbb{Z}^2)). \quad (3.7)$$

By using (3.4) again, we have $L^{\varphi(p)(p^2-1-j_0)} = pm' + 1$ for some integer m' . It follows from $pm'M^{*j_0}\lambda_0 \in \mathbb{Z}^2$ that $L^{\varphi(p)(p^2-1-j_0)}(M^{*j_0}\lambda_0 + \mathbb{Z}^2) = (pm' + 1)(M^{*j_0}\lambda_0 + \mathbb{Z}^2) \subset M^{*j_0}\lambda_0 + \mathbb{Z}^2$. Hence, by (3.6) and (3.7), we have

$$\begin{aligned}\lambda_1 - \lambda_2 &\in L^{\varphi(p)j_0}(M^{*j_0}\lambda_0 + \mathbb{Z}^2) \subset L^{\varphi(p)j_0}(M^{*j_0}\mathcal{Z}_D^2 + \mathbb{Z}^2) \\ &\subset \bigcup_{j=1}^{p^2-1} L^{\varphi(p)j}(M^{*j}\mathcal{Z}_D^2 + \mathbb{Z}^2) \subset \mathcal{Z}(\hat{\mu}_{M,D}).\end{aligned}$$

This shows that $(\Lambda - \Lambda) \setminus \{0\} \subset \mathcal{Z}(\hat{\mu}_{M,D})$. It follows from (2.2) that the elements in \mathcal{E}_Λ are mutually orthogonal, and hence $n^*(\mu_{M,D}) \geq p^2$.

Combining the above with (3.3), we have $n^*(\mu_{M,D}) = p^2$. \square

Now we are ready to prove Theorem 1.1 by Theorem 1.3 and Proposition 2.6.

The proof of Theorem 1.1. By Theorem 1.3, there exist at most p^2 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$. We only need to prove that the number p^2 is the best possible for p is a prime. Let \tilde{M} be the matrix satisfies Proposition 2.6. Since $\emptyset \neq \mathcal{Z}_D^2 \subset \mathring{E}_p^2$, there exists at least one element $\lambda \in \mathcal{Z}_D^2 \subset \mathring{E}_p^2$, and then Proposition 2.6 shows that $\bigcup_{j=1}^{p^2-1} \tilde{M}^j\lambda = \mathring{E}_p^2 \pmod{\mathbb{Z}^2}$. If \tilde{M} is an expanding matrix, then $n^*(\mu_{\tilde{M},D}) = p^2$ by Theorem 1.3. If not, choosing a sufficient large positive integer $N = pm + 1$ such that $\widehat{M} = N\tilde{M}$ is expanding. Note that $pm\tilde{M}^j\lambda \in \mathbb{Z}^2$ for any j . It is easy to see that $\bigcup_{j=1}^{p^2-1} \widehat{M}^j\lambda = \bigcup_{j=1}^{p^2-1} \tilde{M}^j\lambda = \mathring{E}_p^2 \pmod{\mathbb{Z}^2}$, and then $n^*(\mu_{\widehat{M},D}) = p^2$ by using Theorem 1.3 again. \square

The proof of Theorem 1.4. For $p = 2$ or 3 , the conclusion $n^*(\mu_{M,D}) = p^2$ for $M \in \mathfrak{M}_p^{(2)}$ follows from Theorem 1.3. Hence to prove Theorem 1.4, we only need to prove $n^*(\mu_{M,D}) = p$ for $M \in \mathfrak{M}_p^{(1)}$.

Note that $\emptyset \neq \mathcal{Z}_D^2 \subset \mathring{E}_p^2$, there exists at least one element $\lambda \in \mathcal{Z}_D^2 \subset \mathring{E}_p^2$. It is easy to see that $-\lambda \in \mathcal{Z}(m_D)$ because $m_D(-\lambda) = \frac{1}{\#(D)} \sum_{d \in D} e^{2\pi i \langle d, -\lambda \rangle} = \frac{1}{\#(D)} \overline{\left(\sum_{d \in D} e^{2\pi i \langle d, \lambda \rangle} \right)} = 0$. By \mathbb{Z}^2 -periodic of m_D , we have $(\lambda + \mathbb{Z}^2) \cup (-\lambda + \mathbb{Z}^2) \subset \mathcal{Z}(m_D)$. It follows from (2.3) that

$$\mathcal{Z}(\hat{\mu}_{M,D}) \supset \left(\bigcup_{j=1}^{\infty} M^{*j}(\lambda + \mathbb{Z}^2) \right) \cup \left(\bigcup_{j=1}^{\infty} M^{*j}(-\lambda + \mathbb{Z}^2) \right). \quad (3.8)$$

(i) If $p = 2$, then $n^*(\mu_{M,D}) \leq \tilde{P} = \frac{p^2}{2} = 2$ by Theorem 1.3. We now prove $n^*(\mu_{M,D}) \geq 2$. Let $\Lambda = \{0, s_1\}$ with $s_1 = M^*\lambda$ and $\mathcal{E}_\Lambda = \{0, e^{2\pi i \langle s_1, x \rangle}\}$. It is obvious that $(\Lambda - \Lambda) \setminus \{0\} \subset \mathcal{Z}(\hat{\mu}_{M,D})$ since $\pm s_1 \in \mathcal{Z}(\hat{\mu}_{M,D})$. By (2.2), \mathcal{E}_Λ is an orthogonal exponential function set in $L^2(\mu_{M,D})$, and so $n^*(\mu_{M,D}) \geq 2$. This completes the proof of Theorem 1.4 for $p = 2$.

(ii) If $p = 3$, then $n^*(\mu_{M,D}) \leq \tilde{P} = 3$ by Theorem 1.3. Let $\Lambda = \{0, s_1, s_2\}$ with $s_1 = M^*\lambda$ and $s_2 = -M^*\lambda$, and let

$$\mathcal{E}_\Lambda = \{0, e^{2\pi i \langle s_1, x \rangle}, e^{2\pi i \langle s_2, x \rangle}\}.$$

It is easy to see that $\pm s_1, \pm s_2 \in \mathcal{Z}(\hat{\mu}_{M,D})$. By (3.8) and $3\lambda \in \mathbb{Z}^2$, we have $s_1 - s_2 = M^*(2\lambda) = M^*(-\lambda + 3\lambda) \in M^*(-\lambda + \mathbb{Z}^2) \subset \mathcal{Z}(\hat{\mu}_{M,D})$. Similarly, $s_2 - s_1 \in \mathcal{Z}(\hat{\mu}_{M,D})$. These show that $(\Lambda - \Lambda) \setminus \{0\} \subset \mathcal{Z}(\hat{\mu}_{M,D})$. By (2.2), \mathcal{E}_Λ is an orthogonal exponential function set in $L^2(\mu_{M,D})$, hence $n^*(\mu_{M,D}) \geq 3$.

This completes the proof of Theorem 1.4. \square

4. The Sierpinski measures and an example

As an application of the above-mentioned results, in this section we first complete the discussion of planar Sierpinski measures in [12], then we construct an example in which $\mathfrak{M}_p^{(1)} = \emptyset$.

As in [12], by using the residue system of modulo 3, the transposed conjugate M^* of $M = \begin{bmatrix} a & b \\ d & c \end{bmatrix} \in M_2(\mathbb{Z})$ can be expressed in the following way:

$$M^* = \begin{bmatrix} a & d \\ b & c \end{bmatrix} = 3 \begin{bmatrix} l_1 & l_2 \\ l_3 & l_4 \end{bmatrix} + M_\alpha = 3\tilde{M} + M_\alpha, \quad (4.1)$$

where $\tilde{M} \in M_2(\mathbb{Z})$ and the entries of the matrix M_α are 0, 1 or 2. It is easy to see that $M^*\lambda = M_\alpha\lambda \pmod{\mathbb{Z}^2}$ for $\lambda \in E_3^2 \setminus \{0\}$. This shows that if we want to check the matrix M belongs to $\mathfrak{M}_3^{(1)}$ or $\mathfrak{M}_3^{(1)}$, we only need to check M_α . We can easily prove that

there exist only 48 different M_α denoted by $\{M_\alpha\}_{\alpha=1}^{48}$.

Obviously, for each fixed M_α , there are infinitely many expanding matrix $M \in M_2(\mathbb{Z})$ with $\det(M) \notin 3\mathbb{Z}$ such that $M^* - M_\alpha \in M(3\mathbb{Z})$. By using the results in [12], and rearranging the indexes of M_α if necessary, we give the concrete expression of matrix M_α as follows:

$$\begin{aligned}
M_1 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}, M_3 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}, M_4 = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \\
M_5 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, M_6 = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}, M_7 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, M_8 = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \\
M_9 &= \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, M_{10} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, M_{11} = \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}, M_{12} = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}, \\
M_{13} &= \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, M_{14} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, M_{15} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, M_{16} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \\
M_{17} &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, M_{18} = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}, M_{19} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}, M_{20} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \\
M_{21} &= \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}, M_{22} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, M_{23} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, M_{24} = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}, \\
M_{25} &= \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, M_{26} = \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}, M_{27} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, M_{28} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \\
M_{29} &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, M_{30} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}, M_{31} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, M_{32} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \\
M_{33} &= \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}, M_{34} = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}, M_{35} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}, M_{36} = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}, \\
M_{37} &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, M_{38} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, M_{39} = \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}, M_{40} = \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}, \\
M_{41} &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, M_{42} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, M_{43} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}, M_{44} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \\
M_{45} &= \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, M_{46} = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}, M_{47} = \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}, M_{48} = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}.
\end{aligned}$$

For

$$D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \quad (4.2)$$

it is easy to show that $\mathcal{Z}_D^2 = \{(1/3, 2/3)^t, (2/3, 1/3)^t\}$. By a direct calculation or by an application of Propositions 3 – 6 in [12], we have $M_\alpha \in \mathfrak{M}_3^{(1)}$ if $\alpha \in \{1, \dots, 36\}$ and $M_\alpha \in \mathfrak{M}_3^{(2)}$ if $\alpha \in \{37, \dots, 48\}$. Hence the following corollary can be derived from Theorem 1.4 directly.

Corollary 4.1. *Suppose that the self-affine measure $\mu_{M,D}$ is defined by (1.1), where D is given by (4.2), and M is an expanding integer matrix with $\det(M) \notin 3\mathbb{Z}$. Let $n^*(\mu_{M,D})$ be given by (1.7) and $M^* = 3\tilde{M} + M_\alpha$ as (4.1), then*

$$n^*(\mu_{M,D}) = \begin{cases} 3, & \text{if } \alpha \in \{1, \dots, 36\}, \\ 9, & \text{if } \alpha \in \{37, \dots, 48\}. \end{cases}$$

Remark 4.2. In [12], the proof contains a gap and the conclusion is not correct for $\mathfrak{M}_3^{(2)}$. In fact, the third author of this paper assumes that there exist 4 exponential functions $\{e^{2\pi i \langle \lambda_j, x \rangle} : j = 1, 2, 3, 4\}$ being mutually orthogonal in $L^2(\mu_{M,D})$. Then the differences $\lambda_i - \lambda_j \in \mathcal{Z}(\widehat{\mu}_{M,D}) = \bigcup_{j=1}^4 (Z_j \cup \widetilde{Z}_j)$ for $i \neq j$ (see Proposition 2.6 in [12]), and thus the following six differences

$$\lambda_1 - \lambda_2, \lambda_1 - \lambda_3, \lambda_1 - \lambda_4, \lambda_2 - \lambda_3, \lambda_2 - \lambda_4, \lambda_3 - \lambda_4$$

belong to $\mathcal{Z}(\widehat{\mu}_{M,D})$. One can regard the above eight sets $Z_1, Z_2, Z_3, Z_4, \widetilde{Z}_1, \widetilde{Z}_2, \widetilde{Z}_3, \widetilde{Z}_4$ as eight small boxes and let the above six differences belong to these eight small boxes. It has finite many possible distributions, and the following distribution is not considered in [12]:

Z_1	Z_2	Z_3	Z_4	\widetilde{Z}_1	\widetilde{Z}_2	\widetilde{Z}_3	\widetilde{Z}_4
$\lambda_1 - \lambda_2$	$\lambda_1 - \lambda_3$	$\lambda_1 - \lambda_4$	$\lambda_3 - \lambda_2$	$\lambda_2 - \lambda_1$	$\lambda_3 - \lambda_1$	$\lambda_4 - \lambda_1$	$\lambda_2 - \lambda_3$
$\lambda_3 - \lambda_4$	$\lambda_2 - \lambda_4$			$\lambda_4 - \lambda_3$	$\lambda_4 - \lambda_2$		

For the above distribution, we can not get a contradiction by Propositions 2 and 6 in [12]. However, the above Corollary 4.1 gives a complete result.

In the end of this paper, we construct an example to illustrate the case that $\mathfrak{M}_p^{(1)} = \emptyset$.

Example 4.3. Let

$$D_1 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \quad D_2 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}$$

and $D_3 = D_1 + D_2$. Then $n^*(\mu_{M,D_3}) = 9$ for any expanding integer matrix M with $\det(M) \notin 3\mathbb{Z}$, i.e., $\mathfrak{M}_3^{(1)} = \emptyset$.

Proof. Since $D_3 = D_1 + D_2$, we have $m_{D_3}(x) = m_{D_1}(x)m_{D_2}(x)$ and $\mathcal{Z}(m_{D_3}(x)) = \mathcal{Z}(m_{D_1}(x)) \cup \mathcal{Z}(m_{D_2}(x))$. Let $x = (x_1, x_2)^t$, then $m_{D_2}(x) = 1 + e^{2\pi i(3x_1+x_2)} + e^{2\pi i(-x_2)}$. It is well known that $m_{D_2}(x) = 0$ only if

$$\begin{cases} 3x_1 + x_2 = \frac{1}{3} + k_1, \\ -x_2 = \frac{2}{3} + k_2, \end{cases} \quad \text{or} \quad \begin{cases} 3x_1 + x_2 = \frac{2}{3} + k_1, \\ -x_2 = \frac{1}{3} + k_2, \end{cases}$$

where k_1 and k_2 are integers. It is easy to show that

$$\mathcal{Z}_{D_2}^2 = \left\{ \begin{pmatrix} 0 \\ 1/3 \end{pmatrix}, \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix}, \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2/3 \end{pmatrix}, \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}, \begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix} \right\}.$$

Similarly,

$$\mathcal{Z}_{D_1}^2 = \left\{ \begin{pmatrix} 1/3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2/3 \\ 0 \end{pmatrix} \right\}.$$

Hence $\mathcal{Z}_{D_3}^2 = \mathcal{Z}_{D_1}^2 \cup \mathcal{Z}_{D_2}^2 = \mathring{E}_3^2$.

By Proposition 2.2, we have $M^*(\mathcal{Z}_{D_3}^2) = M^*(\dot{E}_3^2) = \dot{E}_3^2(\bmod \mathbb{Z}^2)$ for any expanding matrix M with $\gcd(\det(M), 3) = 1$, and then $\bigcup_{j=1}^8 M^{*j} \mathcal{Z}_{D_3}^2 = \dot{E}_3^2(\bmod \mathbb{Z}^2)$. Hence $M \in \mathfrak{M}_3^{(2)}$ and $n^*(\mu_{M, D_3}) = 9$. \square

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